Note on the stability of plane parallel flows

By D. H. MICHAEL
Pierce Hall, Harvard University†

(Received 1 July 1960)

The subject of this note is the behaviour of three-dimensional small disturbances to plane parallel flows, which have a variation in the direction normal to the plane of mean flow, in relation to two-dimensional disturbances which vary in the plane of mean flow only. It was pointed out by Squire (1933) that, in linearized theory, the disturbance which is neutrally stable at the critical Reynolds number R_c is two-dimensional in form. More recently interest has turned to the question as to which kind of disturbance is most rapidly amplified at a given Reynolds number above the critical. Jungclaus (1957) pointed out that for certain values of R and of the resolved wavelength in the plane of mean flow, three-dimensional disturbances may be more unstable than plane ones. Recently, Watson (1960) has shown further that a two-dimensional disturbance is the one most rapidly amplified in a certain range of R starting from the critical. In this note we take a slightly different view of the problem which enables us to define specifically the upper end of this range of R, when it exists.

It is clear that when a disturbance consists partly of Fourier components which are propagated obliquely to the plane of the mean motion, each such component interacts only with that component of the mean flow in the direction of propagation of the disturbance. We can illustrate this mathematically by choosing a frame of reference which makes a disturbance in any given direction two-dimensional, say a function of x and y, but not of z. In this frame of reference the mean flow will have a component in the z-direction. If we adopt this point of view and consider the disturbance to a plane parallel incompressible flow bounded by planes y = const., we can compute the linearized disturbance equations in which the mean velocities $(U(y), O, W(y))$ are disturbed to $(U+u, v, W+w)$, and in which the small disturbance terms are independent of z, and vary with x and t like $e^{i\alpha(x-ct)}$. In the standard notation the linearized equations for the disturbance are

$$i\alpha(U-c)\,u+v\frac{dU}{dy} = -i\frac{\alpha p}{\rho}+\left(\frac{d^2}{dy^2}-\alpha^2\right)u, \tag{1}$$

$$i\alpha(U-c)\,v = -\frac{1}{\rho}\frac{dp}{dy}+\left(\frac{d^2}{dy^2}-\alpha^2\right)v, \tag{2}$$

$$i\alpha(U-c)\,w+v\frac{dW}{dy} = \left(\frac{d^2}{dy^2}-\alpha^2\right)w, \tag{3}$$

$$i\alpha u+\frac{dv}{dy} = 0. \tag{4}$$

† Now at Department of Mathematics, University College London, W.C. 1.

We see that equations (1), (2) and (4) are not coupled to (3), and with suitable boundary conditions on u and v they provide the characteristic value problem with $U(y)$ the operative component of mean flow. $W(y)$ is only used subsequently in (3), which, when v is established, is an inhomogeneous equation to solve for w with its independent boundary conditions. We may conclude that for a disturbance of wave number α propagated at an angle ϕ to the plane of a mean flow with Reynolds number R, the behaviour is that of a two-dimensional disturbance of wave-number α and reduced Reynolds number $R\cos\phi$. If we know the characteristic curves of constant c_i, where $c = c_r + ic_i$, in the (α, R)-plane, for two-dimensional disturbances of a given velocity profile, this enables us to deduce results simply for three-dimensional disturbances.

It is important in comparing rates of amplification that we should have a correct measure of the amplification rate for different values of α and R. Results of stability calculations are usually given by a plot of curves of constant c_i, in the (α, R)-plane, after these quantities have been made dimensionless. The dimensionless time amplification exponent is αc_i, but the physical exponent is $(V/L)\,\alpha c_i$, where V and L are the representative velocity and length employed, respectively. Normally we think of varying R for a given fluid by changing V with L fixed. In this case we have to take account of the variation of V in our comparisons by writing the time exponent $(\nu/L^2)\,(\alpha c_i R)$. With ν the kinematic viscosity and L fixed, it is clear that a correct dimensionless measure of the time amplification is $\alpha c_i R$. If we were to keep V constant and vary L then the measure would be $\alpha c_i/R$, but for the purpose of comparing growth rates of two- and three-dimensional disturbances in a given apparatus the former case is the relevant one here.

If we continue the discussion in terms of the curves $\alpha c_i R = \text{const.}$ in the (α, R)-plane (where α and c_i are now regarded as dimensionless) the point to decide is whether $\alpha c_i R$ has a maximum as a function of α and R; if it has, some of the curves form closed loops about the stationary point. If we assume this behaviour the curves will be of the form given in figure 1, with L the stationary point, at Reynolds number R_L. (The stationary point L may be accompanied by another stationary point at a larger value of R depending on how α and c_i behave as $R \to \infty$. Such a point will be a saddle point if $R \to \infty$ and $\alpha R \to \infty$ for $c_i > \epsilon > 0$.)

If we construct an ordinate MM' at the Reynolds number of the mean flow, the curve through a point of this line, for given α, describes the amplification of a two-dimensional disturbance of that wave-number. All points in the semi-infinite rectangle $M'MO\alpha$ can be reached by three-dimensional disturbances. Whether a two- or three-dimensional disturbance grows fastest will depend on whether all characteristic $(\alpha c_i R)$-lines in the rectangle cut MM'. There will be two distinct cases depending on whether the Reynolds number of the mean flow is greater or less than R_L. If the ordinate is M_1M_1' say, then the disturbance of most rapid growth is a plane one at the wave-number at which M_1M_1' is a tangent to a characteristic $(\alpha c_i R)$-curve. The amplification associated with this curve is greater than at all points inside the rectangle. Hence for $R_c < R \leqslant R_L$ a plane disturbance is amplified fastest. On the other hand, if we have the ordinate M_2M_2', there will be a family of closed characteristic curves enclosing L which do not intersect M_2M_2' and which provide a higher growth rate than at any point of the line M_2M_2'. Hence for $R > R_L$ a three-dimensional disturbance grows fastest. It may be noted that whereas the maximum amplification rate increases monotonically with R from R_c to R_L, it is constant thereafter, for $R > R_L$, at least up to the next stationary point. In cases where $\alpha c_i R$ does not have a maximum point L we shall know that two-dimensional disturbances will always grow fastest.

Following Watson, we have considered the particular case of plane Poiseuille flow in more detail on the basis of the c_i-curves given by Shen (1954; see also Lin 1955, chap. 3). Figure 2 gives a set of curves $\alpha c_i R$ = const. obtained by interpolation from Shen's curves (the other details of the figure are explained below).

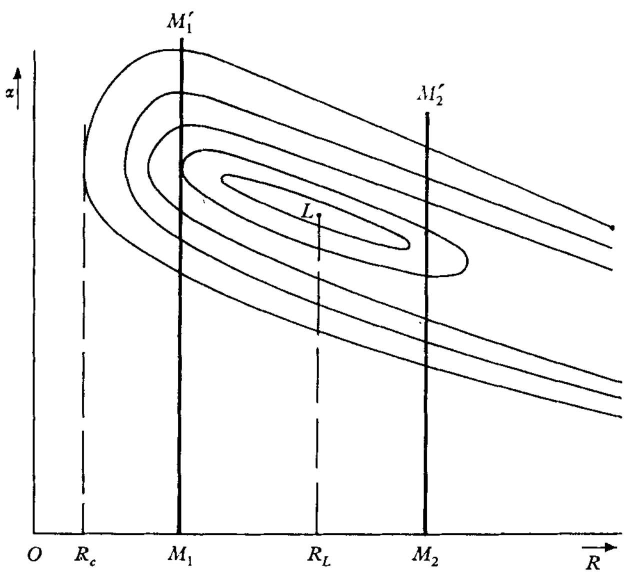

FIGURE 1. Sketch of curves $\alpha c_i R$ = const.

The form of these curves suggests that there is no stationary point L in the range of the Reynolds number covered by the curves, i.e. up to approximately 10^5. This conclusion is in agreement with Watson's conjecture that two-dimensional disturbances grow most rapidly at all values of R.

If we consider Jungclaus's problem in which the resolved component of the wave-number in the plane of mean flow is fixed, the class of disturbance is more restricted. If the resolved wave-number is denoted by β, then the values of α and R for the equivalent plane disturbance are $\beta \sec\phi$ and $R\cos\phi$ respectively. Hence in the (α, R)-plane of figure 1, the locus of three-dimensional disturbances is a branch of a rectangular hyperbola for each pair of values of β and R. In this case, since αR is constant, we can compare amplification rates for different points on any particular hyperbola from the c_i-curves directly, but if we wish to compare points on different hyperbolas the $(\alpha c_i R)$-curves should be used.

Figure 2 also shows a set of curves αR = const. superposed on the $(\alpha c_i R)$-curves for plane Poiseuille flow.

The line of interest here is PQ, which is the locus of points at which the (αR)-curves are tangential to the $(\alpha c_i R)$-curves. If PS is the (αR)-curve which is tangential to the marginal stability curve PT, on which $\alpha c_i R = 0$, then it is clear that for values of β and R chosen in this plane between PQ and PS there are three-dimensional disturbances which are more rapidly amplified than two-

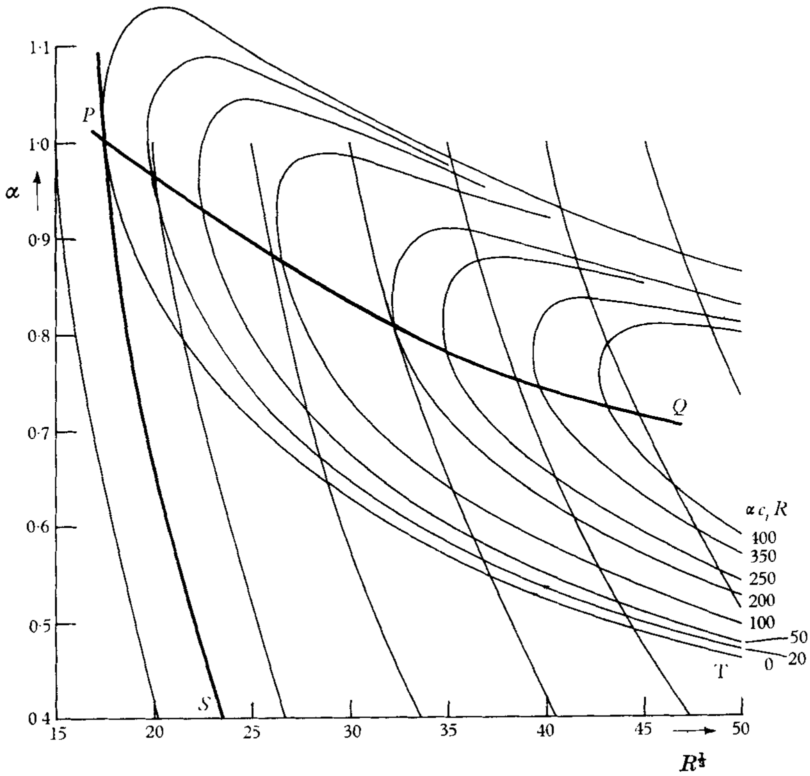

FIGURE 3

FIGURE 2. Curves $\alpha c_i R$ = const. and αR = const. (e.g. the curve PS) for plane Poiseuille flow.

dimensional ones. We also note finally that for values of β and R taken between PS and PT, the plane disturbance, and three-dimensional ones up to a certain value of ϕ, are stable.

The author wishes to acknowledge the financial support received under U.S. Navy Contract No. NONR/1866 (34) during the preparation of this paper.

REFERENCES

JUNGCLAUS, G. 1957 Inst. Fluid Dynamics and Appl. Math., University of Maryland, AFOSR, TN 57–577.

LIN, C. C. 1955 *The Theory of Hydrodynamical Stability*. Cambridge University Press.

SHEN, S. F. 1954 *J. Aero. Sci.* **21**, 62.

SQUIRE, H. B. 1933 *Proc. Roy. Soc.* A, **142**, 621.

WATSON, J. 1960 *Proc. Roy. Soc.* A, **254**, 562.